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ON THE FRATTINI NORMAL EMBEDDABILITY OF PRODUCTS OF p-GROUPS

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ABSTRACT

If H_j is a finite non-abelian p-group with center of order p, for $1 \leqq j \leqq R$, then the direct product of the H_j does not occur as a normal subgroup contained in the Frattini subgroup of any finite p-group. If the Frattini subgroup Φ of a finite p-group G is cyclic or elementary abelian of order p^2, then the centralizer of Φ in G properly contains Φ. Non-embeddability properties of products of groups of order 16 are established.

1. Introduction

Only finite groups will be considered. A group H will be called an FN subgroup (Frattini normal subgroup) of G provided H is normal in G and contained in the Frattini subgroup of G. An FNE group (Frattini normal embeddable group) is one which occurs as an FN subgroup of some group. Similarly a p-FNE group (p-Frattini normal embeddable group) is a group which occurs as an FN subgroup of some p-group. Hobby [5] showed that a non-abelian p-group with cyclic center cannot occur as the Frattini subgroup of a p-group. Bechtell [1] developed fundamental theory relating the Frattini normal embeddability of a p-group G to properties of its $\mathscr{H}$-invariant subgroups, for $\mathscr{H}$ a group of automorphisms of G. The fact that no non-abelian p-group with cyclic center is a p-FNE group is proved in [1], generalizing Hobby's result. A non-embeddability condition for p-groups was given by Hill and Wright [4], and a bound on the nilpotence class of FNE groups was established by Hill and Parker [3]. Hill [2] also places a bound on the exponent of a p-FNE group. Here we shall present another non-embeddability condition (Theorem 6) which yields groups which are not p-FNE groups, but which satisfy the class bound of [3] as well as the exponent bound of [2]. R. W. van der Waall [6] proves the non-embeddability of those groups of order p^4 which were not decided in [3]. In this paper we shall also show that for each of the seven non-2-FNE groups H of

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order 16, $H \times H$ is not a 2-FNE group. The question raised by Bechtell in the closing paragraph of [1] is answered negatively for two families of abelian p-groups.

Our notation is standard. $\Phi(G)$ is the Frattini subgroup of G; $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$ denote the full automorphism group and the inner automorphism group of G, respectively. The ascending central series of a p-group G is denoted by $1 = Z_0(G) \leqq Z_1(G) \leqq \cdots \leqq Z_k(G) = G$, where k is the nilpotence class of G.

2. The basic theorems

The first five theorems regarding p-FNE groups follow from the results in [1], [2] and [3].

THEOREM 1. (Bechtell [1]). *If G is a p-FNE group, $\mathcal{P}$ a p-Sylow subgroup of* $\mathrm{Aut}(G)$, *and N is a $\mathcal{P}$-invariant subgroup of G, then N and G/N are also p-FNE groups.*

THEOREM. 2. (Bechtell [1]). *If G is a p-FNE group, then* $\mathrm{Inn}(G) \leqq \Phi(\mathcal{P})$ *for any p-Sylow subgroup $\mathcal{P}$ of* $\mathrm{Aut}(G)$.

THEOREM 3. (Bechtell [1]). *If G is a non-abelian p-group with cyclic center, then G is not a p-FNE group.*

THEOREM 4. (Hill and Parker [3]). *If for a p-group G there exists $i \neq$ class(G) such that $(Z_i(G): Z_{i-1}(G)) = p$, then G is not an FNE group.*

THEOREM 5. (Hill [2]). *If G is a p-FNE group of order p^n, class k, and exponent p^r, then p^r divides p^{n-k}.*

Essential to the proof of Theorem 6, below, is the following

LEMMA. *Suppose A is an elementary abelian p-group of order p^R, generated by the set $E = \{e_1, e_2, \cdots, e_R\}$. The number of maximal subgroups of A which are disjoint from E is $(p-1)^{R-1}$.*

PROOF. Let $\mathcal{M}$ be the family of maximal subgroups of A, and $\mathcal{N}$ the subfamily of $\mathcal{M}$ consisting of those maximal subgroups which are disjoint from E. A member M of $\mathcal{M}$ is determined by an integer sequence $s = (s_1, s_2, \cdots, s_R)$, $0 \leqq s_j \leqq p-1$, not all $s_j = 0$, in the following way. A group element $a = e_1^{u_1} e_2^{u_2} \cdots e_R^{u_R}$ is in M if and only if $s_1 u_1 + s_2 u_2 + \cdots + s_R u_R \equiv 0$ (modulo p). Moreover, if some $s_j = 0$, then e_j is in M. Therefore, the sequence s determines a member N of $\mathcal{N}$ precisely when all s_j are non-zero. Such an N is uniquely determined by the equivalent sequence $s' = (1, ts_2, ts_3, \cdots, ts_R)$ of non-zero integers, where t is the multiplicative inverse of s_1 (modulo p). There are $(p-1)^{R-1}$ such sequences, so $|\mathcal{N}| = (p-1)^{R-1}$.

Our general non-embeddability result is given in the following

THEOREM 6. *Suppose G is isomorphic to the direct product $H_1 \times H_2 \times \cdots \times H_R$, where each H_j is a non-abelian p-group with center of order p. Then G is not a p-FNE group.*

PROOF. We shall write G as the inner product $H_1H_2\cdots H_R$ of the subgroups H_j, with $Z_1(H_j) = \langle e_j \rangle$. Now $Z_1(G) = \langle e_1, \cdots, e_R \rangle$ is an elementary abelian p-group of order p^R. As in the Lemma we let $\mathcal{N}$ consist of those members of $\mathcal{M}$ which contain no e_j, where $\mathcal{M}$ is the family of maximal subgroups of $Z_1(G)$. Clearly the action of Aut(G) on the family of subgroups of G fixes $Z_1(G)$, and decomposes $\mathcal{M}$ into orbits. Moreover, $\mathcal{N}$ is orbital under this action, that is, $\mathcal{N}$ is the union of orbits in $\mathcal{M}$. We prove this fact by showing that for every M in $\mathcal{M}$, M is in $\mathcal{N}$ if and only if $Z_1(G/M) = Z_1(G)/M$.

Suppose N is in $\mathcal{N}$, and $Z_1(G/N) = W/N$. Clearly $Z_1(G) \leqq W$. If $Z_1(G) < W$, then W contains some element $w = w_1 w_2 \cdots w_R$, w_j in H_j, such that w is not in $Z_1(G)$. Thus, for some i, w_i is not in $Z_1(H_i)$. It follows that $[h_i, w_i] \neq 1$ for some h_i in H_i. However, $[h_i, w_i] = [h_i, w] \in N$, so $[h_i, w_i]$ is a non-identity element of $Z_1(H_i)$. Therefore, $Z_1(H_i) \leqq N$, and in particular e_i is in N, a contradiction. We conclude that $Z_1(G/N) = Z_1(G)/N$ whenever N is in $\mathcal{N}$.

Now suppose M is a maximal subgroup of $Z_1(G)$ which is not in $\mathcal{N}$. M must contain some e_i, so $W \geqq Z_2(H_i)$. Since H_i is non-abelian, we have $W \geqq Z_1(G)Z_2(H_i) > Z_1(G)$. We have shown that $Z_1(G/M) > Z_1(G)/M$ whenever M is not in $\mathcal{N}$.

$\mathcal{N}$ consists of precisely those maximal subgroups N of $Z_1(G)$ for which $Z_1(G/N)$ is of order p. Let $\mathcal{P}$ be a p-Sylow subgroup of Aut(G). From the lemma we know that $|\mathcal{N}| = (p-1)^{R-1} \not\equiv 0 \pmod p$. Thus, there exists some N_1 in $\mathcal{N}$ which is $\mathcal{P}$-invariant. If $R = 1$, then G is not a p-FNE group by Theorem 3. If $R > 1$, then G/N_1 is non-abelian with cyclic center. In this case G is not a p-FNE group by Theorems 1 and 3.

COROLLARY 1. *If $H_1, H_2, \cdots, H_R$ are non-abelian p-groups such that for some fixed i, $(Z_i(H_j): Z_{i-1}(H_j)) = p$, $1 \leqq j \leqq R$, then $H_1 \times H_2 \times \cdots \times H_R$ is not a p-FNE group,*

PROOF. $H_j^+ = H_j/Z_{i-1}(H_j)$ is non-abelian with cyclic center. Letting $G = H_1 \times \cdots \times H_R$, we have $Z_{i-1}(G) = Z_{i-1}(H_1) \times \cdots \times Z_{i-1}(H_R)$. Therefore,

$G/Z_{i-1}(G)$ is isomorphic to $H_1^+ \times \cdots \times H_R^+$, and G is not a p-FNE group by Theorems 1 and 6.

In [3] a group of large class is defined to be a p-group of order p^n and class greater than $n/2$, $n \geqq 2$. A group of large class is not an FNE group ([3], theorem 1). It is not difficult to show that if G is a group of large class, then G is non-abelian and $(Z_i(G): Z_{i-1}(G)) = p$ for some i. Thus, we have

COROLLARY 2. *If G is a group of large class, then no finite product of copies of G is a p-FNE group.*

Suppose G is a group of large class, and let G_* be a finite product of at least two copies of G. G_* is not a p-FNE group even though G_* satisfies the exponent bound of Theorem 5, and G_* is not of large class.

3. Products of groups of order 2^4

Every abelian p-group is a p-FNE group. Let C_m, D_m, and Q_m denote the cyclic, dihedral, and generalized quaternion groups of order $m = 2^n$, respectively. $D_8 \times C_2$ and $Q_8 \times C_2$ both occur as Frattini subgroups of groups of order 2^6. In [3] and [6] it is shown that none of the seven remaining non-abelian groups of order 2^4 is a 2-FNE group. Here we discuss the embeddability of the product groups $H \times H$, where H is of order 2^4. Since $H \times H$ is a 2-FNE group whenever H is, we consider only the seven groups which are not 2-FNE groups. In the following presentations of these seven groups, only non-identity commutators are given.

$$P_r = \langle x, t : x^8 = t^2 = 1, [x, t] = x^r \rangle \qquad \text{for } r = 2, 4, 6,$$

$$Q_{16} = \langle x, t : x^4 = t^2, t^4 = 1, [x, t] = x^6 \rangle,$$

$$R = \langle x, y, z : x^4 = y^2 = z^2 = 1, [y, z] = x^2 \rangle,$$

$$T = \langle x, y : x^4 = y^4 = 1, [x, y] = x^2 \rangle,$$

$$U = \langle x_1, x_2 : x_1^4 = x_2^4 = (x_1^2 x_2^2)^2 = 1, [x_1, x_2] = x_1^2 x_2^2 \rangle.$$

The groups $P_2, P_6 \cong D_{16}$, and Q_{16} are all of class 3. Hence, $P_2 \times P_2, P_6 \times P_6$, and $Q_{16} \times Q_{16}$ are non-2-FNE groups by Corollary 2.

We shall treat the four remaining groups separately. The groups R, T, and U are isomorphic to those of van der Waall's Theorems 1, 2 and 3, respectively, see [6]; but our presentation of U is different.

The following fact will be applied to $P_4 \times P_4$ and to $R \times R$: if μ is a homomorphism from a group G to the symmetric group S_4, then the image of

$\Phi(G)$ under μ is of order 1 or 2. Let M be the kernel of μ, and $\Phi = \Phi(G)$. Since $\Phi(G/M) \geqq \Phi M/M$, and $\Phi\mu$ is isomorphic to $\Phi M/M$, it follows that $\Phi\mu$ lies in the Frattini subgroup of $G\mu$. But the Frattini subgroup of each subgroup of S_4 has order 1 or 2. Therefore, $\Phi\mu$ has order 1 or 2.

Let $H = \langle x_1, t_1, x_2, t_2\rangle \cong \langle x_1, t_1\rangle \times \langle x_2, t_2\rangle \cong P_4 \times P_4$, with the obvious relations holding among the generators. The fifteen involutions in H generate the characteristic subgroup $B = \langle x_1^4, t_1, x_2^4, t_2\rangle$. The action of H on B produces four conjugacy classes of size 1, four of size 2, and one of size 4. Suppose H is an FN subgroup of G. Since $L = \{t_1t_2, x_1^4t_1t_2, x_2^4t_1t_2, x_1^4x_2^4t_1t_2\}$ is the unique class of size 4 in the characteristic subgroup B, L must be a conjugacy class in G. The action of G on L defines a homomorphism from G to $S(L) \cong S_4$. Since H lies in $\Phi(G)$, it follows that the image of H under this homomorphism is of order 1 or 2. This is a contradiction, since H is transitive on the four elements of L. Indeed the action of H on L is generated by x_1 and x_2, and is isomorphic to $C_2 \times C_2$. Therefore, $P_4 \times P_4$ is not an FNE group.

Next let $H = \langle x_1, y_1, z_1, x_2, y_2, z_2\rangle \cong \langle x_1, y_1, z_1\rangle \times \langle x_2, y_2, z_2\rangle \cong R \times R$. The action of H on the set of 63 involutions in H produces three conjugacy classes of size 1, twelve of size 2, and nine of size 4. Suppose H is an FN subgroup of a 2-group G. Since there is an odd number of involution classes of size 4 in H, the action of G must stabilize (set-wise) at least one of these classes. Again we have a homomorphism from G to S_4, under which H must have an image of order 1 or 2. But the action of H must be transitive on the fixed conjugacy class of size 4. It follows that $R \times R$ is not a 2-FNE group. Note that we have not shown that $R \times R$ is not an FNE group. Indeed $|\mathrm{Aut}(R \times R)|$ is divisable by 9, and $\mathrm{Aut}(R \times R)$ is transitive on the 9 involution classes of size 4 in $R \times R$.

At this point we are also able to present a short proof that U (N of Theorem 3 in [6]) is not an FNE group. U contains four copies of C_4, namely $K_1 = \langle x_1\rangle$, $K_2 = \langle x_1x_2^2\rangle$, $K_3 = \langle x_2\rangle$, $K_4 = \langle x_1^2x_2\rangle$. Suppose U is an FN subgroup of a group G. The action of G stabilizes (set-wise) the family $\{K_i\}$ of copies of C_4 in U. This determines the homomorphism from G to S_4. In this case the action of U on $\{K_i\}$ is not transitive. However, this action is still of order 4, since $\phi_{x_1} = (K_1)(K_2)(K_3, K_4)$ and $\phi_{x_2} = (K_1, K_2)(K_3)(K_4)$. Thus, U is not an FNE group.

Now let $H = \langle x_1, y_1, x_2, y_2\rangle \cong \langle x_1, y_1\rangle \times \langle x_2, y_2\rangle \cong T \times T$. Our approach here is to show that $Y = \langle y_1^2, y_2^2\rangle$ is a characteristic subgroup of H. Since $H/Y \cong D_8 \times D_8$, it will then follow from Theorems 1 and 6 that H is not a 2-FNE group. H contains 15 involutions. Let SQ_r denote the set of involutions which have exactly r square roots in H. $SQ_{16} = \{x_1^2, x_2^2, x_1^2x_2^2\}$, $SQ_{32} =$

$\{y_1^2, y_2^2, y_1^2x_2^2, x_1^2y_2^2\}$, $SQ_{64} = \{y_1^2y_2^2\}$, and SQ_0 consists of the remaining seven involutions. Each class SQ_r is stabilized (set-wise) by $\mathrm{Aut}(H)$. Of the six products of pairs of elements from SQ_{32} the characteristic involution $y_1^2y_2^2$ occurs only once. It follows that the pair $\{y_1^2, y_2^2\}$ giving that product must be a characteristic class. Hence, Y is a characteristic subgroup of H.

Finally let $H = \langle x_1, x_2, x_3, x_4\rangle \cong \langle x_1, x_2\rangle \times \langle x_3, x_4\rangle \cong U \times U$. Let $Z_1 = Z_1(H) = \Phi(H) = \langle x_1^2, x_2^2, x_3^2, x_4^2\rangle$, and $X_i = Z_1x_i$, $1 \leqq i \leqq 4$. The conjugacy action of H on itself decomposes the set of 192 elements of order 4 into 32 classes of size 2 and 32 classes of size 4. The union of the four X_i consists of the 32 classes of size 2. Therefore, $\mathrm{Aut}(H)$ stabilizes $X = \bigcup X_i$, and the X_i are blocks of imprimitivity under the action of $\mathrm{Aut}(H)$. Now suppose w_i is in X_i, $1 \leqq i \leqq 4$. Then $\langle w_1, w_2\rangle \cong \langle w_3, w_4\rangle \cong U$, while the other four pairs of w_i generate subgroups isomorphic to $C_4 \times C_4$. Therefore, $\mathrm{Aut}(H)$ also stabilizes the partition $\{\{X_1, X_2\}, \{X_3, X_4\}\}$ of $\{X_i\}$. We can now describe $\mathrm{Aut}(H)$ in terms of its action on X. A member α of $\mathrm{Aut}(H)$ is completely determined by the images under α of the four generators x_i, $1 \leqq i \leqq 4$. For $1 \leqq i, j \leqq 4$, let α_{ij} be the automorphism which maps x_i into $x_ix_j^2$, and x_r into itself, for $r \neq i$. The set $A = \{\alpha_{ij}\}$ is a basis for the automorphism subgroup E which is elementary abelian of order 2^{16}. The action of $\mathrm{Aut}(H)$ on $\{X_i\}$ is some subgroup of D_8, since the partition above is fixed. Letting σ and τ be the members of $\mathrm{Aut}(H)$ defined by the permutations (x_1, x_3, x_2, x_4) and $(x_1, x_2)(x_3)(x_4)$, respectively, we have $D = \langle \sigma, \tau\rangle \cong D_8$. The action of $\mathrm{Aut}(H)$ on $\{X_i\}$ is, in fact, all of D_8. We see that $\mathrm{Aut}(H)$ has order 2^{19}; in particular $\mathrm{Aut}(H) = \langle E, D\rangle$, and $\mathrm{Aut}(H)/E \cong D_8$.

We now examine the action of $\mathrm{Aut}(H)$ on itself, in order to determine $\Phi(\mathrm{Aut}(H))$. The members of A are fixed by E. Therefore, D determines the action of $\mathrm{Aut}(H)$ on A. The action of σ and τ on A is given by

$$\begin{aligned}\phi_\sigma = {} & (\alpha_{11}, \alpha_{33}, \alpha_{22}, \alpha_{44})\,(\alpha_{12}, \alpha_{34}, \alpha_{21}, \alpha_{43}) \\ & (\alpha_{14}, \alpha_{31}, \alpha_{23}, \alpha_{42})(\alpha_{13}, \alpha_{32}, \alpha_{24}, \alpha_{41}),\end{aligned}$$

$$\begin{aligned}\phi_\tau = {} & (\alpha_{11}, \alpha_{22})\,(\alpha_{13}, \alpha_{23})\,(\alpha_{14}, \alpha_{24})\,(\alpha_{31}, \alpha_{32}) \\ & (\alpha_{41}, \alpha_{42})(\alpha_{12})(\alpha_{21})(\alpha_{33})(\alpha_{34})(\alpha_{43})(\alpha_{44}).\end{aligned}$$

It follows that A is decomposed into the three conjugacy classes

$$\begin{aligned}A_1 &= \{\alpha_{11}, \alpha_{22}, \alpha_{33}, \alpha_{44}\}, \\ A_2 &= \{\alpha_{12}, \alpha_{21}, \alpha_{34}, \alpha_{43}\}, \\ A_3 &= \{\alpha_{13}, \alpha_{31}, \alpha_{14}, \alpha_{41}, \alpha_{23}, \alpha_{32}, \alpha_{24}, \alpha_{42}\}.\end{aligned}$$

Clearly $\mathrm{Aut}(H) = \langle \alpha_{11}, \alpha_{12}, \alpha_{13}, \sigma, \tau \rangle$; indeed we shall show that this is a minimal generating set for $\mathrm{Aut}(H)$. Let E_k be the subgroup of index 2 in $\langle A_k \rangle$ consisting of those members of $\langle A_k \rangle$ which are products of evenly many members of A_k, $1 \leqq k \leqq 3$. The group $F = \langle E_1, E_2, E_3 \rangle$ contains all commutators $[\delta, \alpha]$ for δ in D and α in E. Therefore, $\Phi(\mathrm{Aut}(H)) = \langle F, \sigma^2 \rangle$, which is of index 32 in $\mathrm{Aut}(H)$. Hence, $\mathrm{Inn}(G) \nleqq \Phi(\mathrm{Aut}(H))$, since $\phi_{x_1} = \alpha_{21}\alpha_{22}$, which is not a member of $\langle F, \sigma^2 \rangle$. It follows from Theorem 1 that H is not a 2-FNE group. Indeed, since $\mathrm{Aut}(H)$ is itself a 2-group, we see that H is not an FNE group.

4. The centralizer of the Frattini subgroup

Bechtell raises the following question in the closing paragraph of [1]. Suppose F is the Frattini subgroup of a p-group G. Must there exist a p-group G^* such that $\Phi(G^*) \cong F$, and the centralizer of $\Phi(G^*)$ in G^* lies in the center of $\Phi(G^*)$? We answer the question negatively for the cases where F is cyclic or elementary abelian of order p^2.

Suppose $F \cong \Phi = \Phi(G^*)$, where G^* is a p-group. Let E be the centralizer of Φ in G^*. If G^* is abelian, then $E = G^* > \Phi$. We assume that G^* is non-abelian.

First suppose that Φ is cyclic of order p^m. We consider separately the cases p even and p odd. For $p = 2$, Φ is the subgroup of G^* generated by all squares. Hence, x^2 is a generator of Φ for some x in G^*. It follows that $E \geqq \langle \Phi, x \rangle > \Phi$. For p odd, a p-Sylow subgroup $\mathcal{P}$ of $\mathrm{Aut}(\Phi)$ is cyclic of order p^{m-1}. Since G^* is non-abelian, G^*/Φ is elementary abelian of order at least p^2. On the other hand G^*/E can be embedded in the cyclic group $\mathcal{P}$. Since $E \geqq \Phi$, it follows that G^*/E is of order 1 or p. Thus, $E > \Phi$.

Now suppose that Φ is elementary abelian of order p^2. The index of Φ in G^* is at least p^2, since G^* is non-abelian. For p even or odd, a p-Sylow subgroup of $\mathrm{Aut}(\Phi)$ is of order p. Hence, G^*/E is of order 1 or p, so $E > \Phi$.

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